

Finite Temperature Effects for Massive Fields in D -dimensional Rindler-like Spaces

Andrei A. Bytsenko¹

State Technical University, St. Petersburg 195251, Russia

Guido Cognola² and Sergio Zerbini³

Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare,
Gruppo Collegato di Trento, Italia

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Abstract: *The first quantum corrections to the free energy for massive fields in D -dimensional space-times of the form $\mathbb{R} \times \mathbb{R}^+ \times \mathcal{M}^{N-1}$, where $D = N + 1$ and \mathcal{M}^{N-1} is a constant curvature manifold, is investigated by means of the ζ -function regularization. It is suggested that the nature of the divergences, which are present in the thermodynamical quantities, might be better understood making use of the conformal related optical metric and associated techniques. The general form of the horizon divergences of the free energy is obtained as a function of free energy densities of fields having negative square masses (absence of the gap in the Laplace operator spectrum) on ultrastatic manifolds with hyperbolic spatial section H^{N-2n} and of the Seeley-DeWitt coefficients of the Laplace operator on the manifold \mathcal{M}^{N-1} . Furthermore, recurrence relations are found relating higher and lower dimensions. The cases of Rindler space, where $\mathcal{M}^{N-1} = \mathbb{R}^{N-1}$ and very massive D -dimensional black holes, where $\mathcal{M}^{N-1} = S^{N-1}$ are treated as examples. The renormalization of the internal energy is also discussed.*

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1 Introduction

Recently there has been a renewed interest in the physics of black holes. Several issues like the interpretation of the Bekenstein-Hawking classical formula for the black hole entropy, the puzzle of loss of information in the black hole evaporation and the interpretation of the Hawking temperature have been discussed (see, for example, the review [1]). Furthermore in many papers, it has been pointed out that it should be desirable to have the usual statistical interpretation of the black hole entropy as the number of the gravitational states at the horizon and to try to understand the dynamical origin of the black hole entropy (see, for example [2]).

¹email: root@fmf.stu.spb.su (subject: Prof. A.A. Bytsenko)

²e-mail: cognola@science.unitn.it

³e-mail: zerbini@science.unitn.it

On general grounds, the density of levels as a function of the mass M , of a D -dimensional black hole should read [3] (for details, see Sec. 6.3)

$$\Omega(M) \simeq C_D(M) \exp \left(\frac{4\pi\hat{G}_D}{D-2} M^{\frac{D-2}{D-3}} \right). \quad (1.1)$$

where $C_D(M)$ is a quantum prefactor and \hat{G}_D is related to the generalized Newton constant (see Eq. (6.9)). For a 4-dimensional black hole we have [4]

$$\Omega(M) \simeq C_4(M) \exp(4\pi G M^2), \quad (1.2)$$

$4\pi G M^2$ being the Bekenstein-Hawking classical entropy [5, 6, 7]. As 't Hooft has pointed out, the prefactor $C_4(M)$ (the first quantum correction to the classical result), which is usually computable for quantum fields or extended objects (such as strings or p-branes) in an ultrastatic space-time background, turns out to be divergent. This prefactor may be regarded as the contribution associated with the first quantum correction to the classical free energy. Several different methods have been used in dealing with such divergences, for example "the brick wall method" [4, 8, 9], the conical singularity method [10, 11, 12, 13, 14, 15], critically discussed in [16, 17] and the related "entanglement entropy method" [18, 19, 20].

The horizon divergences have also been associated with the information loss issue of black holes [4, 8] and their physical origin, for quantum fields [21, 22] or strings [23, 24, 25, 26], may be described by the following simple considerations.

In a D -dimensional static space-time with horizons, the equivalence principle implies that a system in thermal equilibrium has a local Tolman temperature given by $T(x) = T/\sqrt{-g_{00}(x)}$, T being the asymptotic temperature. Roughly speaking, near the canonical horizon (this means that the quantity g_{00} has simple zeros), a static space-time may be regarded as a Rindler-like space-time. We will show that, if one denotes by ρ the proper distance from the horizon, one gets for the Tolman temperature $T(\rho) = T/\rho$. As a consequence, the total entropy for a quantum bosonic gas reads (omitting a multiplicative constant)

$$S \equiv \int d\mathbf{x} \int_0^\infty T(\rho)^{D-1} d\rho \simeq A T^{D-1} \int_0^\infty \rho^{-D+1} d\rho, \quad (1.3)$$

where A is the integral on the transverse coordinates \mathbf{x} , namely the area of the horizon. The latter integral is clearly divergent. Introducing a horizon cutoff parameter ε we may rewrite it as

$$\begin{aligned} S &\simeq A T \int_\varepsilon^\Lambda \rho^{-1} d\rho \simeq A T \ln \frac{\Lambda}{\varepsilon}, & \text{for } D = 2 \text{ } \Lambda \text{ infrared cutoff} \\ S &\simeq A T^{D-1} \int_\varepsilon^\infty \rho^{-D+1} d\rho \simeq \frac{A T^{D-1}}{(D-2)\varepsilon^{D-2}}, & \text{for } D > 2. \end{aligned} \quad (1.4)$$

For the sake of generality, we write down the asymptotic high temperature expansion for the entropy of a quantum gas on a $D = N + 1$ -dimensional static space-time defined by the metric

$$ds^2 = g_{00}(\mathbf{x})(dx^0)^2 + g_{ij}(\mathbf{x})dx^i dx^j, \quad \mathbf{x} = \{x^j\}, \quad i, j = 1, \dots, N, \quad (1.5)$$

$g(\mathbf{x})$ denoting its determinant. Again the equivalence principle leads to

$$S \simeq T^N \int \left(\frac{g(\mathbf{x})}{g_{00}(\mathbf{x})} \right)^{-N/2} dx^N. \quad (1.6)$$

As a consequence, the horizon divergences depend on the nature of the poles of the integrand $(g/g_{00})^{-N/2}$. In general, for extremal black holes, where g_{00} has higher order zeroes, the divergences are much more severe than in the non extremal case (see for example Refs. [27, 17, 28]).

These considerations suggest the use of the optical metric $\bar{g}_{\mu\nu} = g_{\mu\nu}/g_{00}$, conformally related to the original one, in order to investigate these issues. It is our opinion that the conformal transformation techniques are particularly suitable for studying finite temperature effects for fields in space-times with horizons and here we would like to present some examples of computation. This method has already been appeared for example in Refs. [29, 30, 31] and has been recently used in the horizon divergence problems in Refs. [22, 2, 16, 17, 32, 33]. See also [34], where the same result is obtained with a different approach.

One of the purposes of this paper is to implement this idea in the case of massive fields in D -dimensional Rindler-like space-times, we are going to introduce. Let us consider static space-times admitting canonical horizons and having the topology of the form $\mathbb{R} \times \mathbb{R}^+ \times \mathcal{M}^{N-1}$. The metric reads

$$ds^2 = -\frac{b^2 \rho^2}{r_H^2} dx_0^2 + d\rho^2 + d\sigma_{N-1}^2, \quad (1.7)$$

where r_H is a dimensional constant, b a constant factor and $d\sigma_{N-1}^2$ the spatial metric related to the $N-1$ -dimensional manifold \mathcal{M}^{N-1} . If $\mathcal{M}^{N-1} \equiv \mathbb{R}^{N-1}$, $b = 1$ and $r_H = 1/a$, a being a constant acceleration, one has to deal with the Rindler space-time. Quantum fields in such a case have been considered in many places, see for example Refs. [35, 36, 37, 38, 39]. If $\mathcal{M}^{N-1} \equiv S^{N-1}$, $b = (D-3)/2$ and r_H is the Schwarzschild radius of a black hole, then we shall show that one is dealing with a space-time which approximates, near the horizon and in the large mass limit, a D -dimensional black hole (see Sec. 6.2).

It is well known that space-times with canonical horizons admit a distinguished temperature, the (Unruh) Hawking temperature. There are several ways to compute it, one of the simplest makes use of the relation with the related surface gravity. The other one consists in imposing the absence of conical singularities in the Euclidean continuation of the space-time itself [7]. For the metric (1.7), one obtains

$$\beta_H = \frac{2\pi r_H}{b}. \quad (1.8)$$

One can arrive at the same result working without using the Euclidean continuation method, but making use of the principle of local definiteness in quantum field theory [36, 40]. Note that these methods are no longer equivalent when one is dealing with extremal black holes [41]. It is also important to stress that the variable ρ defined by the metric in Eq. (1.7) has the meaning of radial proper distance between the horizon and a point outside it and so, the divergences of thermodynamical quantities are automatically expressed in an invariant way.

The contents of the paper are the following. In Sec. 2, a review of the necessary conformal transformation techniques is presented. In Sec. 3, we consider a Laplace-type operator defined on a class of $D = N+1$ -dimensional space-times, whose spatial sections have metrics conformally related to $\mathcal{M}^N = \mathbb{R}^+ \times \mathcal{M}^{N-1}$. Since in general \mathcal{M}^N is a non compact manifold, the Laplace operator has a continuum spectrum and a general form for the Plancherel measure, which is the analogue of the degeneracy in the case of discrete spectrum, is presented. The measure is used in Sec. 4 in order to obtain a useful form for the trace of the heat kernel, which is necessary for the derivation of the free energy, which we derive in Sec. 5. It is pointed out that in the Rindler case, the spatial section of the conformally related space-time turns out to be an N -dimensional hyperbolic manifold. In Sec. 6 some applications to the statistical mechanics of a scalar field in D -dimensional Rindler and black hole space-times are presented and the divergences of the first quantum corrections to free energy and entropy are given. Finally, we end with some conclusions in Sec. 7 and with a resume of heat kernel, ζ -function and free energy on constant curvature manifolds in the Appendix.

2 Conformal transformation techniques and optical manifold

In this section we shall briefly summarize the method of conformal transformations using ζ -function regularization [29, 42, 43, 44]. These techniques permit to compute all physical quantities in an ultrastatic manifold (called the optical manifold [45]) and, at the end of calculations, transform back them to a static one. This method is particularly useful in dealing with finite temperature effects for quantum fields, since these effects can be easily investigated in ultrastatic space-times.

To start with, we consider a non self-interacting scalar field on a $D = N + 1$ -dimensional static space-time defined by the metric (1.6), i.e.

$$ds^2 = g_{00}(\mathbf{x})(dx^0)^2 + g_{ij}(\mathbf{x})dx^i dx^j, \quad \mathbf{x} = \{x^j\}, \quad i, j = 1, \dots, N. \quad (2.1)$$

The one-loop partition function is given by (we perform the Wick rotation $x_0 = -i\tau$, thus all the differential operator one is dealing with will be elliptic)

$$Z = \int d[\phi] \exp \left(-\frac{1}{2} \int \phi L_D \phi d^D x \right), \quad (2.2)$$

where ϕ is a scalar density of weight $-1/2$ and the operator L_D has the form

$$L_D = -\Delta_D^g + m^2 + \xi R^g. \quad (2.3)$$

Here m (the mass) and ξ are arbitrary parameters, while Δ_D^g and R^g are respectively the Laplace-Beltrami operator and the scalar curvature of the manifold in the original metric g .

The ultrastatic metric $\bar{g}_{\mu\nu}$ can be related to the static one by the conformal transformation

$$\bar{g}_{\mu\nu}(\mathbf{x}) = e^{2\sigma(\mathbf{x})} g_{\mu\nu}(\mathbf{x}), \quad (2.4)$$

with $\sigma(\mathbf{x}) = -\frac{1}{2} \ln g_{00}$. In this manner, $\bar{g}_{00} = 1$ and $\bar{g}_{ij} = g_{ij}/g_{00}$ (optical metric). Recalling that by a conformal transformation (we remind that ϕ is a scalar density)

$$\begin{aligned} R^{\bar{g}} &= e^{-2\sigma} [R^g - 2(D-1)\Delta_D^g \sigma - (D-1)(D-2)g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma], \\ \bar{\phi} &= e^\sigma \phi, \\ \Delta_D^{\bar{g}} \bar{\phi} &= e^{-\sigma} \left[\Delta_D^g - \frac{D-2}{2} \Delta_D^g \sigma - \frac{(D-2)^2}{4} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right] \phi \\ &= e^{-\sigma} \left[\Delta_D^g + \xi_D (e^{2\sigma} R^{\bar{g}} - R^g) \right] \phi, \end{aligned} \quad (2.5)$$

one obtains

$$L_D \phi = e^\sigma \left\{ -\Delta_D^{\bar{g}} + \xi_D R^{\bar{g}} + e^{-2\sigma} [m^2 + (\xi - \xi_D) R^g] \right\} \bar{\phi}, \quad (2.6)$$

where $\xi_D = (D-2)/4(D-1)$ is the conformal invariant factor. From the latter equation we have $\phi L_D \phi = \bar{\phi} \bar{L}_D \bar{\phi}$, where, by definition

$$\bar{L}_D = e^{-\sigma} L_D e^{-\sigma} = -\Delta_D^{\bar{g}} + \xi_D R^{\bar{g}} + e^{-2\sigma} [m^2 + (\xi - \xi_D) R^g]. \quad (2.7)$$

This means that action $\bar{I} = I$ by definition. Note that classical conformal invariance requires the action to be invariant in form, that is $\bar{I} = I[\bar{\phi}, \bar{g}]$, as to say $\bar{L}_D = L_D$. As is well known, this happens only for conformally coupled massless fields ($\xi = \xi_D$).

For the one-loop partition function we have

$$\bar{Z} = J[g, \bar{g}] Z, \quad (2.8)$$

where $J[g, \bar{g}]$ is the Jacobian of the conformal transformation. Such a Jacobian can be computed for any infinitesimal conformal transformation [42]. To this aim it is convenient to introduce a family of continuous conformal transformations

$$g_{\mu\nu}^q = e^{2q\sigma} g_{\mu\nu} = e^{2(q-1)\sigma} \bar{g}_{\mu\nu}, \quad \sqrt{g^q} \equiv \sqrt{|\det g_{\mu\nu}^q|} = e^{Dq\sigma} \sqrt{g}, \quad (2.9)$$

in such a way that the metric is $g_{\mu\nu}$ or $\bar{g}_{\mu\nu}$ according to whether $q = 0$ or $q = 1$ respectively. Then one gets

$$\ln J[g_q, g_{q+\delta q}] = \ln \frac{Z_{q+\delta q}}{Z_q} = \frac{\delta q}{(4\pi)^{D/2}} \int k_D(x|L_D^q) \sigma(x) \sqrt{g^q} d^D x, \quad (2.10)$$

where $k_D(x|L_D^q)$, is the Seeley-DeWitt coefficient, which in the case of conformal invariant theories, is proportional to the trace anomaly. In general, one has the asymptotic expansion

$$\text{Tr } e^{-tL_D} \simeq \sum_n K_n(L_D) t^{\frac{n-D}{2}}, \quad K_n(L_D) = \frac{1}{(4\pi)^{D/2}} \int_{\mathcal{M}} k_n(x|L_D) \sqrt{g} d^D x. \quad (2.11)$$

If the manifold is without boundary then $K_n = 0$ for any odd n . The heat kernel coefficients are computable and depend on invariant quantities built up with curvature (field strength) and their derivatives (see, for example, [46]).

The Jacobian for a finite transformation can be obtained from Eq. (2.10) by an elementary integration in q [42]. In particular we have

$$\ln J[g, \bar{g}] = \frac{1}{(4\pi)^{D/2}} \int_0^1 dq \int k_D(x|L_D^q) \sigma(x) \sqrt{g^q} d^D x, \quad (2.12)$$

and finally, making use of the ζ -function regularization, one has

$$\ln Z = \ln \bar{Z} - \ln J[g, \bar{g}] = \frac{1}{2} \zeta'(0|\bar{L}_D \ell^2) - \ln J[g, \bar{g}], \quad (2.13)$$

where ℓ is an arbitrary parameter necessary to adjust the dimensions and ζ' represents the derivative with respect to s of the ζ -function $\zeta(s|\bar{L}_D \ell^2)$ related to the operator \bar{L}_D , which is given by Eq. (2.7).

The same analysis can be easily extended to the finite temperature case [43]. In fact we recall that for a scalar field in thermal equilibrium at finite temperature $T = 1/\beta$ in an ultrastatic space-time, the corresponding partition function \bar{Z}_β may be obtained, within the path integral approach, simply by Wick rotation $\tau = ix^0$ and imposing a β periodicity in τ for the field $\bar{\phi}(\tau, x^i)$ ($i = 1, \dots, N$, $N = D - 1$) [47, 48, 49, 50]. In this way, in the one loop approximation, one has

$$\bar{Z}_\beta = \int_{\bar{\phi}(\tau, x^i) = \bar{\phi}(\tau + \beta, x^i)} d[\bar{\phi}] \exp \left(- \int_0^\beta d\tau \int \bar{\phi} \bar{L}_D \bar{\phi} d^N x \right). \quad (2.14)$$

in which

$$\bar{L}_D = -\partial_\tau^2 - \bar{\Delta}_N + \xi_D R^{\bar{g}} + e^{-2\sigma} [m^2 + (\xi - \xi_D) R^g] = -\partial_\tau^2 + \bar{L}_N. \quad (2.15)$$

Since the space-time is ultrastatic, by means of the ζ -function regularization again one easily obtain [51]

$$\begin{aligned} \ln \bar{Z}_\beta &= -\frac{\beta}{2} \left[\text{PP } \zeta(-\tfrac{1}{2}|\bar{L}_N) + (2 - 2 \ln 2\ell) \text{Res } \zeta(-\tfrac{1}{2}|\bar{L}_N) \right] \\ &\quad + \lim_{s \rightarrow 0} \frac{d}{ds} \frac{\beta}{\sqrt{4\pi} \Gamma(s)} \sum_{n=1}^{\infty} \int_0^\infty t^{s-3/2} e^{-n^2 \beta^2 / 4t} \text{Tr } e^{-t\bar{L}_N} dt. \end{aligned} \quad (2.16)$$

where PP and Res stand for the principal part and the residue of the function. The free energy is related to the canonical partition function by means of equation

$$F(\beta) = -\frac{1}{\beta} \ln Z_\beta = -\frac{1}{\beta} (\ln \bar{Z}_\beta - \ln J[g, \bar{g}]) = F_0 + F_\beta, \quad (2.17)$$

where F_0 represents the vacuum energy, which is given by the first term in Eq. (2.16), while F_β represents the temperature dependent part (statistical sum). The entropy and the internal energy of the system are given by the usual thermodynamical formulae

$$S_\beta = \beta^2 \partial_\beta F(\beta) = \beta^2 \partial_\beta F_\beta. \quad (2.18)$$

$$U_\beta = \beta \partial_\beta F(\beta) + F_\beta = -e \partial_\beta \ln Z_\beta. \quad (2.19)$$

In a similar way, one can consider spinor fields. It is sufficient to make use of the following formal identity [44]

$$F_f(\beta) \equiv 2F_{2\beta} - F_\beta, \quad (2.20)$$

where on the r.h.s. the spinor quantities are left understood in the formal expression of the bosonic free energy. As a consequence, the horizon divergences of the bosonic sector cannot be compensated by the corresponding fermionic ones.

3 Spectral function for rank 1 Riemannian spaces conformally related to $\mathcal{M}^N = \mathbb{R}^+ \times \mathcal{M}^{N-1}$

In many interesting physical cases, the Euclidean optical metric may be written in the form (see Eq. (1.7) and Sec. 6)

$$d\bar{s}^2 = d\tau^2 + \frac{r_H^2}{\rho^2} (d\rho^2 + d\sigma_{N-1}^2), \quad (3.1)$$

where $d\tau = bdx^0$, r_H being a characteristic length (for example the horizon radius), $(r_H/\rho)^2$ the conformal factor and $d\sigma_{N-1}^2$ the metric of a $N-1$ -dimensional manifold.

Here we derive the spectral measure of the operator \bar{L}_N , as defined by Eq. (2.15), acting on scalars in the spatial section of the manifold defined by the metric Eq. (3.1). Using such equations (for convenience now we put $r_H = 1$; in this way all quantities are dimensionless; the dimensions will be easily restored at the end of calculations) we easily obtain

$$\begin{aligned} dV &= \rho^{-N} d\rho dV_{N-1}, \\ \bar{L}_N &= -\Delta_N^{\bar{g}} - \varrho_N^2 + C\rho^2, \\ \Delta_N^{\bar{g}} &= \rho^2 \partial_\rho^2 - (N-2)\rho \partial_\rho + \rho^2 \Delta_{N-1}, \end{aligned} \quad (3.2)$$

where Δ_{N-1} is the Laplace-Beltrami operator on the manifold \mathcal{M}^{N-1} and dV_{N-1} its invariant measure. We have also set $\varrho_N = (N-1)/2$ and $C = m^2 + \xi R^g$. It should be noted the appearance of an effective "tachionic" mass $-\varrho_N^2$, which has important consequences on the structure of the ζ -function related to the operator \bar{L}_N .

In order to study the quantum properties of matter fields defined on this ultrastatic manifold, it is sufficient to investigate the kernel of the operator $e^{-t\bar{L}_N}$. To this aim, we will search for the spectral resolution of the elliptic operator \bar{L}_N . Let $\Psi_{r\alpha}(x)$ be its eigenfunctions, namely

$$\bar{L}_N \Psi_{r\alpha}(x) = \lambda_r^2 \Psi_{r\alpha}(x) = \lambda_r^2 f_\alpha(\mathbf{x}) \phi_{r\alpha}(\rho), \quad (3.3)$$

where $f_\alpha(\mathbf{x})$ are the (normalized) eigenfunctions of the reduced operator $L_{N-1} = -\Delta_{N-1} + C$ with eigenvalues λ_α^2 . Note that we assume C to be constant. This means that we restrict ourselves to consider only manifolds \mathcal{M}^D with constant scalar curvature, or alternatively minimally coupled fields. Moreover, to avoid null eigenvalues we suppose $C > 0$, but the results can be easily extended to the case $C = 0$. Note that the spectrum of L_{N-1} could also be continuum.

The differential equation which determines the continuum spectrum turns out to be

$$\left[\rho^2 \partial_\rho^2 - (N-2)\rho \partial_\rho - \rho^2 \lambda_\alpha^2 + \varrho_N^2 + \lambda_r^2 \right] \phi_{r\alpha}(\rho) = 0. \quad (3.4)$$

The only solutions of the latter equation with the correct decay properties at infinity are the Bessel functions of imaginary argument $K_{ir}(\rho\lambda_\alpha)$ with $\lambda_r^2 = r^2$ (if $C = 0$ the operator L_{N-1} has a zero mode and gives other solutions to Eq. (3.4)). Thus we have

$$\phi_{r\alpha}(\rho) = \rho^{\frac{N-1}{2}} K_{ir}(\rho\lambda_\alpha). \quad (3.5)$$

If we interpret in the sense of the distribution the following inner product

$$(\Psi_{r\alpha}, \Psi_{r'\alpha'}) = \int_0^\infty \frac{d\rho}{\rho^N} \int dV_{N-1} \Psi_{r\alpha}^*(x) \Psi_{r'\alpha'}(x), \quad (3.6)$$

we have the normalization condition

$$(\Psi_{r\alpha}, \Psi_{r'\alpha'}) = \delta_{\alpha\alpha'} \frac{\delta(r-r')}{\mu(r)}, \quad (3.7)$$

where $\mu(r)$ is the spectral measure associated with the continuum spectrum. Thus, for the heat kernel of any suitable function $f(\bar{L}_N)$ we may write

$$\langle x | f(\bar{L}_N) | x' \rangle = \int_0^\infty dr \mu(r) f(r^2) \sum_\alpha \Psi_{r\alpha}^*(x') \Psi_{r\alpha}(x). \quad (3.8)$$

The measure $\mu(r)$ may be determined in the following standard way (Harish-Chandra's method [52]), which makes use of the asymptotic behaviour of the Mac Donald functions at the origin. From Eq. (3.4) and its complex conjugate and making use of Eq. (3.5) one arrives at

$$(\lambda_r^2 - \lambda_{r'}^2) (\phi_{r\alpha}, \phi_{r'\alpha'}) = \lim_{\rho \rightarrow 0} \rho^{-(N-2)} (\partial_\rho \phi_{r\alpha}^* \phi_{r'\alpha'} - \phi_{r\alpha}^* \partial_\rho \phi_{r'\alpha'}) . \quad (3.9)$$

By means of Eqs. (3.5) and (3.7) we get

$$\frac{\delta(r-r')}{\mu(r)} = \lim_{u \rightarrow 0} \frac{u}{r^2 - r'^2} [\partial_u K_{ir}^*(u) K_{ir'}(u) - K_{ir}^*(u) \partial_u K_{ir'}(u)], \quad (3.10)$$

where again, the limit has to be understood in the sense of distributions. Recalling that for $u \rightarrow 0$

$$K_{ir}(u) \sim \frac{1}{2} \left[\Gamma(-ir) \left(\frac{u}{2} \right)^{ir} + \Gamma(ir) \left(\frac{u}{2} \right)^{-ir} \right], \quad ir \notin \mathbb{Z} \quad (3.11)$$

and

$$\lim_{u \rightarrow 0} \frac{u^{\pm ix}}{x} = \mp \pi i \delta(x), \quad (3.12)$$

one finally has

$$\mu(r) = \frac{2}{\pi |\Gamma(ir)|^2} = \frac{2}{\pi^2} r \sinh \pi r, \quad (3.13)$$

which is in agreement with the 2-dimensional Kontorovich-Lebedev inversion formula [53].

Since our aim is to evaluate the trace of functions of \bar{L}_N using Eq. (2.11), in particular $\text{Tr} \exp(-t\bar{L}_N)$, it is convenient to make the sum over α , introducing the total spectral measure

$$\mu_{\bar{L}_N}(r, x) = \mu(r) \rho^{N-1} \sum_{\alpha} |f_{\alpha}(\mathbf{x}) K_{ir}(\rho \lambda_{\alpha})|^2 \quad (3.14)$$

and integrate on the manifold defining

$$\mu_I(r) = \int_{\mathcal{M}^N} \mu_{\bar{L}_N}(r, x) dV = \mu(r) \int_0^{\infty} \sum_{\alpha} |K_{ir}(\rho \lambda_{\alpha})|^2 \frac{d\rho}{\rho}, \quad (3.15)$$

where we have used the normalization properties of f_{α} . In this way, for any suitable function $f(\bar{L}_N)$ we have

$$\text{Tr} f(\bar{L}_N) = \int_0^{\infty} f(r^2) \mu_I(r) dr. \quad (3.16)$$

As a simple application of Eq. (3.14) let us consider a massless scalar field in a D -dimensional Rindler space-time. In this case the optical spatial section turns out to be the hyperbolic space H^N and the measure $\mu_{\bar{L}_N}$ should not depend on x , since one is dealing with a homogeneous space and it should coincide with the known Plancherel measure. For this case $\mathcal{M}^{N-1} = \mathbb{R}^{N-1}$ and moreover $C = 0$. The reduced operator $L_{N-1} = -\Delta_{N-1}$ has a continuum spectrum, the eigenvalues being $k^2 = \mathbf{k} \cdot \mathbf{k}$ and the corresponding eigenfunctions $f_{\mathbf{k}} = (2\pi)^{-(N-1)/2} \exp(i\mathbf{k} \cdot \mathbf{x})$. As a consequence

$$\begin{aligned} \Phi_N(r) \equiv \mu_{\bar{L}_N}(r) &= \mu(r) \frac{\Omega_{N-2} \rho^{N-1}}{(2\pi)^{N-1}} \int_0^{\infty} |K_{ir}(\rho k)|^2 k^{N-2} dk \\ &= \frac{2}{(4\pi)^{N/2} \Gamma(N/2)} \frac{|\Gamma(ir + \varrho_N)|^2}{|\Gamma(ir)|^2}, \end{aligned} \quad (3.17)$$

Ω_N being the volume of the N -dimensional sphere. Of course, Eq. (3.17) is the correct Plancherel measure of the Laplace operator in H^N [54].

4 The heat kernel for massive fields

Here we derive a general expression for $\mu_I(r)$ by making use of Eq. (3.15) and then derive the trace of the heat kernel, which is needed for the construction of the partition function according to Eq. (2.16). Now we use the Mellin-Barnes representation [55]

$$K_{ir}^2(\rho \lambda_{\alpha}) = \frac{1}{4i\sqrt{\pi}} \int_{\text{Re } z > 1} \frac{\Gamma(z + ir)\Gamma(z - ir)\Gamma(z)}{\Gamma(z + 1/2)} \rho^{-2z} \lambda_{\alpha}^{-2z} dz \quad (4.1)$$

and observe that, for $\text{Re } z > (N-1)/2$, the sum over α can be done and gives

$$\sum_{\alpha} \lambda_{\alpha}^{-2z} = \zeta(z|L_{N-1}). \quad (4.2)$$

In integrating over ρ , one has to pay attention to the fact that the result is formally divergent. For this aim we introduce a horizon cutoff parameter ε and, when possible, we take the limit $\varepsilon \rightarrow 0$. Then we get

$$\mu_I(r) = \frac{\mu(r)}{8i\sqrt{\pi}} \int_{\text{Re } z = c > (N-1)/2} \frac{\Gamma(z + ir)\Gamma(z - ir)\Gamma(z)\zeta(z|L_{N-1})}{z\Gamma(z + 1/2)\varepsilon^{2z}} dz. \quad (4.3)$$

The integration over z can be done since the meromorphic structure of Γ - and ζ -functions are known. In fact, we have [56]

$$\Gamma(z)\zeta(z|L_{N-1}) = \sum_{n=0}^{\infty} \frac{K_n(L_{N-1})}{z - \frac{N-1-n}{2}} + J_{N-1}(z), \quad (4.4)$$

where J_{N-1} is an analytic function. Since the manifold \mathcal{M}^{N-1} has no boundary, all K_n with odd n are vanishing.

To make the integral we consider the rectangular contour $\Gamma \equiv \{\text{Re } z = c, \text{Im } z = a, \text{Re } z = -c, \text{Im } z = -a\}$ and observe that the two horizontal paths $\text{Im } z = \pm a$ give a vanishing contribution in the limit $a \rightarrow \infty$, as well as the path $\text{Re } z = -c$ in the limit $\varepsilon \rightarrow 0$. Also the poles in the strip $-c < \text{Re } z < 0$ give a vanishing contribution as soon as $\varepsilon \rightarrow 0$. Then we have to take into consideration only the poles of the integrand in Eq. (4.3) in the half-plane $\text{Re } z \geq 0$. Such a function has simple poles at the points $z = 0$, $z = -n \pm ir$ and $z = (N-1-n)/2$ ($n \geq 0$). If D is even, that is N is odd, $z = 0$ is a double pole. It is clear that all poles with $\text{Re } z > 0$ give rise to divergences, the number of them depending on N , while the poles at $z = 0$ and $z = \pm ir$ give rise to finite contributions. As a result one obtains

$$\begin{aligned} \mu_I(r) = & \sum_{n=0}^{\left[\frac{N-2}{2}\right]} \frac{K_{2n}(L_{N-1}) \Phi_{N-2n}(r)}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2}\right)^{\frac{N-1-2n}{2}} \\ & + \frac{\zeta'(0|L_{N-1})}{2\pi} + \frac{\zeta(0|L_{N-1})}{2\pi} \left[\psi(ir) + \psi(-ir) - 2 \ln \frac{\varepsilon}{2} - \pi \delta(r) \right], \quad (4.5) \end{aligned}$$

where $\left[\frac{N-2}{2}\right]$ is the integer part of the number $\frac{N-2}{2}$, $\psi(z)$ the logarithmic derivative of Γ and $\delta(r)$ the usual Dirac δ -function. Note that for even N , $\zeta(0|L_{N-1})$ is vanishing and so the last term in the latter equation disappears.

Now the trace of the heat kernel can be computed by using Eq. (3.16) with $f(r^2) = \exp(-tr^2)$. We write it in the form

$$\begin{aligned} \text{Tr } e^{-t\tilde{L}_N} = & \sum_{n=0}^{\frac{N-3}{2}} \frac{K_{2n}(L_{N-1}) K(t| - \Delta_{H^{N-2n}} - \varrho_{N-2n}^2)}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2}\right)^{\frac{N-1-2n}{2}} \\ & + \frac{1}{4\sqrt{\pi t}} \left[\zeta'(0|L_{N-1}) - 2\zeta(0|L_{N-1}) \ln \frac{\varepsilon}{2} \right] \\ & - \frac{\zeta(0|L_{N-1})}{4} + \frac{\zeta(0|L_{N-1})}{2\pi} \int_{-\infty}^{\infty} \psi(ir) e^{-tr^2} dr, \quad (4.6) \end{aligned}$$

$$\begin{aligned} \text{Tr } e^{-t\tilde{L}_N} = & \sum_{n=0}^{\frac{N-2}{2}} \frac{K_{2n}(L_{N-1}) K(t| - \Delta_{H^{N-2n}} - \varrho_{N-2n}^2)}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2}\right)^{\frac{N-1-2n}{2}} \\ & + \frac{1}{4\sqrt{\pi t}} \zeta'(0|L_{N-1}), \quad (4.7) \end{aligned}$$

valid for odd and even N respectively. Here by $K(t| - \Delta_{H^{N-2n}} - \varrho_{N-2n}^2)$ we indicate the diagonal heat kernel of a Laplace-like operator on H^{N-2n} . Of course, it does not depend on the coordinates since hyperbolic manifolds are homogeneous. Such a kernel is known in any dimension [51] (see the Appendix).

As in the previous section, as a simple application of Eq. (4.5), we again consider a massless scalar field in the D -dimensional Rindler space-time. We have $K_0(L_{N-1}) = (4\pi)^{-\frac{N-1}{2}} V_{N-1}$, $K_n = 0$ for $n > 0$ and $\zeta(z|L_{N-1}) = 0$ for $z < (N-1)/2$. Here V_{N-1} is the volume of the manifold \mathcal{M}^{N-1} (infinite transverse area). Then, using Eq. (3.17), we immediately obtain

$$\mu_I(r) = \Phi_N(r) V_\varepsilon, \quad V_\varepsilon = \frac{V_{N-1} \varepsilon^{-(N-1)}}{N-1}, \quad (4.8)$$

which is the integral version of Eq. (3.17). Here V_ε may be considered as the volume of H^N .

5 The thermodynamical quantities

Now it is quite straightforward to obtain the partition function and then all the others thermodynamical quantities by means of Eqs. (2.16) and (2.17). Since the vacuum energy has been extensively studied in many papers, here we concentrate our attention on the temperature dependent part of the free energy (statistical sum) $F_\beta = \bar{F}_\beta = -\ln \bar{Z}_\beta / \beta$. Using Eqs. (4.6) and (4.7) we get

$$F_\beta^{\text{even } D} = \sum_{n=0}^{\frac{N-3}{2}} \frac{K_{2n}(L_{N-1}) \mathcal{F}_{N-2n}^\beta}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2} \right)^{\frac{N-1-2n}{2}} - \frac{\pi}{12\beta^2} \left[\zeta'(0|L_{N-1}) - 2\zeta(0|L_{N-1}) \ln \frac{\varepsilon}{2} \right] - \frac{\zeta(0|L_{N-1})}{4} \frac{\ln \beta}{\beta} + \frac{\zeta(0|L_{N-1})}{2\pi\beta} \int_0^\infty [\psi(ir) + \psi(-ir)][1 - e^{-\beta r}] dr, \quad (5.1)$$

$$F_\beta^{\text{odd } D} = \sum_{n=0}^{\frac{N-2}{2}} \frac{K_{2n}(L_{N-1}) \mathcal{F}_{N-2n}^\beta}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2} \right)^{\frac{N-1-2n}{2}} - \frac{\pi}{12\beta^2} \zeta'(0|L_{N-1}), \quad (5.2)$$

where \mathcal{F}_{N-2n}^β indicates the free energy density for a scalar field with (negative) square mass $-\varrho_{N-2n}^2$ on an ultrastatic manifold with hyperbolic H^{N-2n} spatial section, which has been studied in detail in Ref. [51] and is given in the Appendix.

Some remarks on Eqs. (5.1) and (5.2) are in order. First of all, it has to be noted that the parameter β is the inverse of the physical temperature only if $b = 1$. More generally, before to interpret β^{-1} as the temperature in Eqs. (5.1) and (5.2), one has to make the substitution $\beta \rightarrow b\beta$. The reason is due to the fact that, in order to write the metric (1.7) in the form (3.1), we have changed the time coordinate according to $\tau = bx_0$.

Independently on the manifold \mathcal{M}^{N-1} , we see that the (non renormalized) free energy has a leading divergence of the kind $\varepsilon^{-(D-2)}$ proportional to the transverse area V_{D-2} , since K_0 and \mathcal{F}_N^β are always non vanishing. More generally, one has $\left[\frac{D-1}{2} \right]$ divergences of the kind $\varepsilon^{-(D-2-2n)}$ (depending on the manifold and the operator L_{N-1}) and, for even D , also a possible logarithmic divergence. All these divergences are also present in the expressions of internal energy and entropy and their expressions can be obtained by means of Eqs. (2.19) and (2.18). For example the internal energy reads

$$U_\beta^{\text{even } D} = \sum_{n=0}^{\frac{N-3}{2}} \frac{K_{2n}(L_{N-1}) \mathcal{U}_{N-2n}^\beta}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2} \right)^{\frac{N-1-2n}{2}} + \frac{\pi}{12\beta^2} \left[\zeta'(0|L_{N-1}) - 2\zeta(0|L_{N-1}) \ln \frac{\varepsilon}{2} \right] + \frac{\zeta(0|L_{N-1})}{4\beta} - \frac{\zeta(0|L_{N-1})}{2\pi} \int_0^\infty r [\psi(ir) + \psi(-ir)] e^{-\beta r} dr + U_0(\varepsilon), \quad (5.3)$$

$$U_\beta^{\text{odd } D} = \sum_{n=0}^{\frac{N-2}{2}} \frac{K_{2n}(L_{N-1}) \mathcal{U}_{N-2n}^\beta}{N-1-2n} \left(\frac{4\pi}{\varepsilon^2} \right)^{\frac{N-1-2n}{2}} + \frac{\pi}{12\beta^2} \zeta'(0|L_{N-1}) + U_0(\varepsilon), \quad (5.4)$$

where \mathcal{U}_{N-2n}^β indicates the free energy density for a scalar field with (negative) square mass $-\varrho_{N-2n}^2$ on an ultrastatic manifold with hyperbolic H^{N-2n} spatial section and $U_0(\varepsilon)$ is the vacuum energy.

With regard to the internal energy, we have at disposal a renormalization procedure, which is well understood for $D = 4$. In fact, in Rindler and black hole space-times it is known that the renormalized stress-energy tensor is finite at the horizon in the Hartle-Hawking state [57, 31], corresponding to the temperature $\beta = \beta_H$. This is equivalent to write

$$U(\beta)^{ren} = U_\varepsilon(\beta) - U_\varepsilon(\beta_H) + \text{finite part}, \quad (5.5)$$

where $U_\varepsilon(\beta)$ is the divergent part of the internal energy and it may be read off the Eqs. (5.3) and (5.4). Thus, the divergences are present in the expression of the renormalized internal energy, but only for some particular value of β , say β_H (β_U). For example, in the case of Rindler space-time, such a value is $\beta_U = 2\pi a^{-1}$, the Unruh temperature (here a is the acceleration), while in the 4-dimensional black hole background one has $\beta_H = 8\pi MG$, the Hawking temperature.

In the general case, we may use of the same renormalization procedure. Note, however, that the corresponding renormalized partition function, free energy and entropy remain divergent also at the distinguished temperature $\beta = \beta_H$.

6 Some physical applications

As simple physical applications of the general formulae derived in Sec. 5, here we consider the cases in which \mathcal{M}^{N-1} is a homogeneous manifold with constant scalar curvature κ . Of course we have the three possibilities $\mathcal{M}^{N-1} \equiv \mathbb{R}^{N-1}$ ($\kappa = 0$), $\mathcal{M}^{N-1} \equiv S^{N-1}$ ($\kappa > 0$) and finally $\mathcal{M}^{N-1} \equiv H^{N-1}$ ($\kappa < 0$), but here we only consider in more detail the first two cases. The first one corresponds to the conformal treatment of the D -dimensional Rindler space-time, while the second appears when one studies the physics of black holes near the horizon. For $D = 4$, this case has been studied in Ref. [58]).

6.1 Statistical mechanics for massive fields in the Rindler D -dimensional space-time

As we have already observed, after a conformal transformation, the spatial section of the Rindler space-time is of the kind considered in the paper. For this special case, the curvature of \mathcal{M}^{N-1} is vanishing ($k = 0$) and so one easily has

$$K_{2n}(L_{N-1}) = \frac{(-m^2)^n}{n!} \frac{V_{N-1}}{(4\pi)^{\frac{N-1}{2}}}, \quad \zeta(z|L_{N-1}) = \frac{V_{N-1}\Gamma(z - \frac{N-1}{2})}{(4\pi)^{\frac{N-1}{2}}\Gamma(z)} m^{N-1-2z}, \quad (6.1)$$

where $C = m^2$ has been put since Rindler is a flat manifold. Now, using Eqs. (5.1) and (5.2) together with the two equations above, we obtain

$$\begin{aligned} F_\beta^{Rind} &= \sum_{n=0}^{\frac{N-3}{2}} \frac{V_{N-1} \varepsilon^{-(N-1-2n)}}{(N-1-2n)n!} \left(-\frac{m^2}{4\pi}\right)^n \mathcal{F}_{N-2n}^\beta \\ &\quad - \frac{\pi}{12\beta^2} \frac{V_{N-1}}{\Gamma(\frac{N+1}{2})} \left[\gamma + \psi\left(\frac{N+1}{2}\right) - \ln \frac{m^2 \varepsilon^2}{4} \right] \left(-\frac{m^2}{4\pi}\right)^{\frac{N-1}{2}} \\ &\quad + \frac{V_{N-1}}{\Gamma(\frac{N+1}{2})} \left(-\frac{m^2}{4\pi}\right)^{\frac{N-1}{2}} \left[-\frac{\ln \beta}{4\beta} + \frac{1}{2\pi\beta} \int_0^\infty [\psi(ir) + \psi(-ir)][1 - e^{-\beta r}] dr \right], \quad (6.2) \end{aligned}$$

$$\begin{aligned} F_\beta^{Rind} &= \sum_{n=0}^{\frac{N-2}{2}} \frac{V_{N-1} \varepsilon^{-(N-1-2n)}}{(N-1-2n)n!} \left(-\frac{m^2}{4\pi}\right)^n \mathcal{F}_{N-2n}^\beta \\ &\quad - \frac{\pi}{12\beta^2} V_{N-1} \Gamma\left(-\frac{N-1}{2}\right) \left(-\frac{m^2}{4\pi}\right)^{\frac{N-1}{2}}, \quad (6.3) \end{aligned}$$

valid for even and odd D -dimension respectively. In Eq. (6.2) γ is the Euler constant. The functions \mathcal{F}_{N-2n}^β can be computed using Eqs. (A.5), (A.8) and (A.12) in the Appendix.

For example, when $D = 4$, using Eq. (A.13), the result is

$$F_\beta^{Rind} = -\frac{A\pi^2}{180\beta^4\epsilon^2} + \frac{Am^2(1 - \ln \frac{m^2\epsilon^2}{4})}{48\beta^2} + \frac{Am^2}{4\pi} \left[\frac{\ln \beta}{4\beta} - \frac{1}{2\pi\beta} \int_0^\infty [\psi(ir) + \psi(-ir)][1 - e^{-\beta r}] dr \right], \quad (6.4)$$

where the transverse area $A = V_2$ has been introduced to compare the latter formula with well known results (see for example Refs. [8, 15, 59, 60]). There is agreement in the massless case, but not in the massive case, where we also obtain a finite contribution. We conclude this section with some remarks on renormalization. As we have seen above, in our formalism, massive scalar fields in Rindler space-time can be easily treated, because the optical spatial section turns out to be the hyperbolic space H^3 and the harmonic analysis on such a manifold is well known. The formulae are particularly simple in the massless case. For example, in 4-dimensions, the total free energy may be chosen in the form

$$F^{ren}(\beta) = -\frac{A}{45(8\pi)^2\epsilon^2} \left[\left(\frac{\beta_U}{\beta} \right)^4 + 3 \right], \quad (6.5)$$

where $\beta_U = 2\pi$ is the Unruh temperature ($a = 1$). As a consequence, the entropy turns out to be

$$S_\beta = \frac{8\pi^2 A}{45\epsilon^2 \beta^3} \quad (6.6)$$

and it diverges for every finite β , but is zero at zero temperature (the Fulling-Rindler state), which is correct, since we are dealing with a pure state. At $\beta = \beta_U$, corresponding to the Minkowski vacuum, we have a divergent entropy proportional to the area, regardless of the fact that the Minkowski vacuum is a pure state. This is also to be expected, since an uniformly accelerated observer cannot observe the whole Minkowski space-time. Finally with this renormalization prescription, the internal energy should read

$$U^{ren}(\beta) = \frac{A}{15(8\pi)^2\epsilon^2} \left[\left(\frac{\beta_U}{\beta} \right)^4 - 1 \right] \quad (6.7)$$

and this is vanishing and a fortiori finite at $\beta = \beta_U$, as it should be. Furthermore, at $\beta = \infty$, namely in the Fulling-Rindler vacuum, it is in agreement with the result obtained in Ref. [61].

6.2 Statistical mechanics for massive fields in a D -dimensional black hole background

Here we consider in more detail the case in which $\mathcal{M}^{N-1} = S^{N-1}$. To justify this choice from a physical view point, first of all we show that, near the horizon, a D -dimensional black hole may be approximated by a manifold of this kind and so, the thermodynamics can be derived by using the formulae of Sec. 5.

The static metric describing a D -dimensional Schwarzschild black hole (we assume $D > 3$) reads [62]

$$ds^2 = - \left[1 - \left(\frac{r_H}{r} \right)^{D-3} \right] dx_0^2 + \left[1 - \left(\frac{r_H}{r} \right)^{D-3} \right]^{-1} dr^2 + r^2 d\Omega_{D-2}, \quad (6.8)$$

where we are using polar coordinates, r being the radial one and $d\Omega_{D-2}$ the $D-2$ -dimensional spherical unit metric. The horizon radius is given by

$$r_H = \hat{G}_D M^{\frac{1}{D-3}}, \quad \hat{G}_D = \left[\frac{2\pi^{\frac{D-3}{2}} G_D}{(D-2)\Gamma(\frac{D-1}{2})} \right]^{\frac{1}{D-3}}, \quad (6.9)$$

M being the mass of the black hole and G_D the generalized Newton constant. The associated Hawking temperature reads $\beta_H = 4\pi r_H/(D-3)$. The corresponding Bekenstein-Hawking entropy may be computed by making use of

$$\beta_H = \frac{\partial S_H}{\partial M}. \quad (6.10)$$

Thus we have

$$S_H = 4\pi \frac{\hat{G}_D}{D-2} M^{\frac{D-2}{D-3}} \quad (6.11)$$

From now on, we put $r_H = 1$. It may be convenient to redefine the radial Schwarzschild coordinate $r = r(\rho)$ by means of the implicit relation

$$\begin{aligned} \rho^2 &= \frac{4}{D-3} \left[e^{r-1} \exp \int \frac{dr}{r^{D-3}-1} \right]^{\frac{1}{D-3}}, \\ &\sim \frac{2}{D-3} (r-1) e^{\frac{(D-2)(r-1)}{2}} + \dots, \end{aligned} \quad (6.12)$$

and time $x_0 = x'_0/b$, $b = (D-3)/2$ in order to have $g_{00} = \rho^2 + O(\rho^4)$. In the new set of coordinates we have

$$ds^2 = -\frac{1-r^{3-D}(\rho)}{b^2} dx_0'^2 + \frac{1-r^{3-D}(\rho)}{b^2 \rho^2} d\rho^2 + r^2(\rho) d\Omega_{D-2}, \quad (6.13)$$

and finally the optical metric reads

$$d\bar{s}^2 = -dx_0'^2 + \frac{1}{\rho^2} [d\rho^2 + G(\rho) d\Omega_{D-2}], \quad (6.14)$$

where we have set

$$G(\rho) = \frac{(br\rho)^2}{1-r^{3-D}} = 1 + O(\rho^2). \quad (6.15)$$

From the latter equation we see that, near the horizon $\rho = 0$, we can set $G(\rho) = 1$ and so the optical metric assumes the form considered in previous Sections. In this approximation the manifold \mathcal{M}^{N-1} becomes the unit sphere S^{N-1} . We have

$$d\bar{s}^2 \simeq -dx_0'^2 + \frac{1}{\rho^2} [d\rho^2 + d\Omega_{D-2}]. \quad (6.16)$$

Such a metric can be considered as an approximation of the one of the black hole in Eq. (6.14) in the sense that, near the horizon, the geodesics are essentially the same for both the metrics. The metric (6.16) can be related to a manifold with curvature $R^{\bar{g}} = -(D-1)(D-2) + O(\rho^2)$, then, according to Eq. (2.7), the relevant operator becomes

$$\bar{L}_N = -\Delta_N^{\bar{g}} - \varrho_N^2 + C\rho^2 + O(\rho^4), \quad (6.17)$$

where now C is a positive constant, which takes into account of mass and curvature contributions to this order. Note that since for the original manifold $R^g = 0$, ξ does not appear in the formulae.

This effectively happens if we approximate the metric after the optical transformation has been done. More simply, one can put $\xi = \xi_D$ in Eq. (2.7).

The discussion for arbitrary $D = N + 1$ is quite involved even though it may be done, since the ζ -functions of the Laplace-Beltrami operators on S^{N-1} are known (see the Appendix).

As a more explicit example, now we consider a scalar field in a 4-dimensional Schwarzschild background. Using these techniques, such a case has been considered in Ref. [58], where we refer the interested reader for more details. We have $r_H = 2MG$, $b = 1/2$,

$$\rho = 2(r - 1)^{\frac{1}{2}} e^{(r-1)/2}, \quad (6.18)$$

and

$$r = 1 + \frac{\rho^2}{4} - \frac{\rho^4}{16} + O(\rho^6). \quad (6.19)$$

Then, according to Eq. (6.17), the relevant operator becomes

$$\bar{L}_3 = -\bar{\Delta}_3 - 1 + C\rho^2, \quad (6.20)$$

where $C = m^2 + 1/3$ takes into account of the curvature $R^g = -6 + 2\rho^2$ of the optical manifold.

Now, directly using Eqs. (2.16), (2.17), (5.1) and (A.13), after the replacement $\beta \rightarrow \beta/2$ due to the redefinition of the Schwarzschild time (remember that $b = 1/2$), for the total free energy we obtain

$$\begin{aligned} F^{bh}(\beta) = & -Aj_\varepsilon + \frac{1}{4}\zeta(-\tfrac{1}{2}|\bar{L}_3) - \frac{2\pi^2 A}{45\varepsilon^2\beta^4} - \frac{A}{12\beta^2} \left[\frac{\zeta'(0|L_2)}{2} - \zeta(0|L_2) \ln \frac{\varepsilon}{2} \right] \\ & - \frac{A\zeta(0|L_2)}{16\pi\beta} \ln \frac{\beta}{2} + \frac{A\zeta(0|L_2)}{8\pi^2\beta} \int_0^\infty \ln(1 - e^{-\beta r/2}) [\psi(ir) + \psi(-ir)] dr, \end{aligned} \quad (6.21)$$

where we have written the Jacobian contribution to the partition function due to the conformal transformation in the form $A\beta j_\varepsilon$, and now $A = 4\pi r_H^2$ is the transverse area of the black hole. The ζ -function related to the operator L_2 on the sphere is given in the Appendix by Eq. (A.18) with $C = m^2 + 1/3$, then $\zeta(0|L_2) = m^2$.

The leading divergence, due to the optical volume, is proportional to the horizon area [4], but in contrast with the Rindler case, a logarithmic divergence is also present, similar to the one found in Refs. [12, 34]. This is a feature of even dimensions (see Sec. 5).

Let us briefly discuss the renormalization of the internal energy in this particular case. We recall that one needs a renormalization in order to remove the vacuum divergences. These divergences, as well as the Jacobian conformal factor, do not contribute to the entropy. However the situation presented here is complicated by the presence of horizon divergences, controlled by the cutoff parameter ε . In the 4-dimensional Schwarzschild space-time, it is known that the renormalized stress-energy tensor is well defined at the horizon in the Hartle-Hawking state [57, 31], which in our formalism corresponds to the Hawking temperature $\beta = \beta_H$. The renormalized internal energy reads (the dots stay for finite contributions at the horizon, which we do not write down because their value depend on the approximation made)

$$U^{ren}(\beta) = \frac{A}{30(8\pi)^2\varepsilon^2} \left[\left(\frac{\beta_H}{\beta} \right)^4 - 1 \right] - \frac{A}{3(8\pi)^2} \ln \varepsilon \left[\left(\frac{\beta_H}{\beta} \right)^2 - 1 \right] + \dots, \quad (6.22)$$

which has no divergences for $\beta = \beta_H$, the Hawking temperature, while the entropy

$$S_\beta = \frac{8\pi^2 A}{45\varepsilon^2\beta^3} - \frac{A \ln \varepsilon}{6\beta} + \dots, \quad (6.23)$$

also for $\beta = \beta_H$ contains the well known divergent term proportional to the horizon area [4] and, according to Ref. [12], a logarithmic divergence too. Eq. (4.4) is vanishing in the Boulware vacuum corresponding to $\beta = \infty$.

From this renormalization procedure we get for the renormalized black hole free energy

$$\begin{aligned}
F^{ren}(\beta) = & -\frac{A}{90(8\pi)^2\varepsilon^2} \left[\left(\frac{\beta_H}{\beta} \right)^4 + 3 \right] + \frac{A}{3(8\pi)^2} \ln \varepsilon \left[\left(\frac{\beta_H}{\beta} \right)^2 + 1 \right] \\
& -\frac{A}{12\beta^2} \left[\frac{\zeta'(0|L_2)}{2} + m^2 \ln 2 \right] - \frac{Am^2}{16\pi\beta} \ln \frac{\beta}{2} \\
& + \frac{Am^2}{8\pi^2\beta} \int_0^\infty \ln(1 - e^{-\beta r/2}) [\psi(ir) + \psi(-ir)] dr. \quad (6.24)
\end{aligned}$$

In the general case, the discussion is quite similar and it can be performed by using Eqs. (5.1) or (5.2) with the replacement $\beta \rightarrow (D-3)\beta/2$.

7 Conclusions

In this paper the first quantum corrections to the thermodynamic quantities of fields in a D -dimensional Rindler-like space have been investigated making use of conformal transformation techniques and ζ -function regularization. In this way, we have worked within the so called optical manifold, which is ultrastatic, and the use of finite temperature methods is quite straightforward.

The general form of the horizon divergences of the free energy has been obtained as a function of free energy densities of fields having negative square masses (absence of the gap in the Laplace operator spectrum) on ultrastatic manifolds with hyperbolic spatial section H^{N-2n} and of the Seeley-DeWitt coefficients $K_{2n}(L_{N-1})$ of the Laplace operator on \mathcal{M}^{N-1} . Since there exists recurrence relations for free energy densities (see the Appendix), it is sufficient to study the cases $D = 3$ and $D = 4$ ($D = 4$ and $D = 5$ for applications to black holes). The leading divergence can be seen to be given by the volume of the spatial section of the optical manifold. A finite contribution is also obtained and this depends on $\zeta(0|L_{N-1})$ and on its first derivative. For $D = 4$, our results are consistent with the ones obtained with brick wall, path-integral and canonical methods [9, 17, 32].

With regard to physical applications, we have used the general results on finite temperature field theory in order to investigate the quantum corrections to the Bekenstein-Hawking entropy for massive fields in a large mass black hole background. This approach gives rise to a leading divergence for the entropy similar to the one obtained for the Rindler case background, but in this case other divergent contributions are present and their structure depend on the dimension of the space-time considered. Here we have shown how it is possible to get the general form valid for an arbitrary dimension and we have explicitly considered the case $D = 4$.

We also would like to mention the results obtained in Ref. [63], where the contributions to the 4-dimensional black hole entropy due to modes located inside and near the horizon have been evaluated using a new invariant statistical mechanical definition for the black hole entropy. The finite contributions, namely the ones independent on the horizon cutoff, are compatible with our results.

As far as the horizon divergences are concerned we recall that they may be interpreted physically in terms of the infinite gravitational redshift existing between the spatial infinity, where one measures the generic equilibrium temperature and the horizon, which is classically inaccessible for the Schwarzschild external observer. Furthermore, we have argued that they are absent in the internal energy at the Unruh-Hawking temperature. However, they remain in the entropy and in the other thermodynamical quantities, as soon as one assumes the validity of the usual thermodynamical relations. For $D = 4$, a possible way to deal with such divergences has been suggested in Refs. [4, 63, 64], where it has been argued that the quantum fluctuations at the horizon might provide a natural cutoff. In particular, choosing the horizon cutoff parameter of the order of the Planck length ($\varepsilon^2 \sim G$), the leading "divergence", evaluated at the Hawking

temperature, turns out to be of the form of the the "classical" Bekenstein-Hawking entropy. This seems a reasonable assumption, because we have worked within the fixed background approximation. However one should remark that other terms are present, giving contributions which violate the area law. A more elaborate discussion for $D = 4$ can be found in Ref. [65]. Alternatively, one may try to relate the horizon divergences to the ultraviolet divergences of quantum gravity, thus arriving at the theory of superstring propagating in a curved space-time [8] or at the renormalization group approach [66].

Finally, we mention that there has been the proposal to absorb the horizon divergences, at least for $D = 2, 3, 4$, by making use of the standard ultraviolet gravitational constant renormalization [8, 67, 68, 69, 70]. This proposal is essentially based on the use of Euclidean section with a conical singularity and associated heat kernel expansion. The problematic issue consisting in dealing with a finite temperature theory in a non ultrastatic space-time is solved working within a non vanishing conical singularity and interpreting the deficit angle of the Euclidean compactified time as the inverse of the equilibrium temperature (the absence of the conical singularity gives the correct Hawking temperature). However, the resulting partition function has, apparently, a wrong dependence on this "temperature". Furthermore, it seems that the only divergences present are the usual ultraviolet ones associated with the definition of the partition function as determinant of an elliptic operator. These divergences are then absorbed in the gravitational constant renormalization. However, the naive use of ζ -function regularization should get rid off these ultraviolet divergences. Thus it seems to exist a disagreement between this approach and our approach based on the conformal transformation techniques. It is our opinion that this disagreement might depend on a non commutative property present in the evaluation of heat kernel trace on a cone. It should be interesting to elucidate this issue.

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A Heat kernel, ζ -function and free energy in constant curvature spaces

To start with, let us write down a representation for the statistical sum which may be useful for investigating the high temperature expansion. For detailed derivations we refer the reader to Ref. [51], where all material of this Appendix can be found. It reads

$$\mathcal{F}_N^\beta = -\frac{1}{\pi i} \int_{\text{Re } z=c} dz \zeta(z)_R \Gamma(z-1) \zeta\left(\frac{z-1}{2} | A_N\right) \beta^{-z}, \quad (\text{A.1})$$

where $A_N = -\Delta_N + \alpha^2 + \kappa \varrho_N^2$, acting on fields in H^N or S_N , with α a constant which may go to zero.

A.1 Hyperbolic manifolds

Here we only report the equations concerning heat kernel, ζ -function and free energy for scalars fields in hyperbolic manifolds, we need in the paper.

The measure for the Laplace-Beltrami operator acting on scalar fields in H^N satisfy the recurrence relations

$$\Phi_{N+2}(r) = \frac{\varrho_N^2 + r^2}{2\pi N} \Phi_N(r), \quad (\text{A.2})$$

which permit to derive recurrence formulae for all others quantities one is interested in. In particular, for the operator A_N (here $\kappa = -1$) we have

$$K(t|A_{N+2}) = -\frac{1}{2\pi N} \left[\partial_t + \alpha^2 + \kappa \varrho_N^2 \right] K(t|A_N), \quad (\text{A.3})$$

$$\tilde{\zeta}(s|A_{N+2}) = -\frac{1}{2\pi N} \left[(\alpha^2 + \kappa \varrho_N^2) \tilde{\zeta}(s|A_N) - \tilde{\zeta}(s-1|A_N) \right], \quad (\text{A.4})$$

where $K(t|A_N)$ and $\tilde{\zeta}(s|A_N)$ are densities. Finally, for the free energy density, one gets

$$\mathcal{F}_{N+2}^\beta = -\frac{1}{2\pi N} \left[(\alpha^2 + \kappa \varrho_N^2) \mathcal{F}_N^\beta - \tilde{\zeta}'(-1|A_N) \right]. \quad (\text{A.5})$$

Using equations above, one obtains the desired quantities for fields in H^N , starting from H^3 or H^2 according to whether N is odd or even. For the space H^3 we have

$$K(t|A_3) = \frac{e^{-t\alpha^2}}{(4\pi t)^{3/2}}, \quad (\text{A.6})$$

$$\tilde{\zeta}(s|A_3) = \frac{\Gamma(s - \frac{3}{2})}{(4\pi)^{\frac{3}{2}} \Gamma(s)} \alpha^{3-2s}, \quad (\text{A.7})$$

$$\mathcal{F}_3^\beta = -\frac{1}{\pi i} \int_{\text{Re } z=c} dz \zeta_R(z) \Gamma(z-1) \frac{\Gamma(\frac{z}{2}-2)}{\Gamma(\frac{z-1}{2})} \beta^{-z} \frac{\alpha^{4-z}}{(4\pi)^{3/2}}, \quad (\text{A.8})$$

or equivalently

$$\mathcal{F}_3^\beta = -\frac{\alpha^4}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta\alpha)}{(n\beta\alpha)^2}, \quad (\text{A.9})$$

while for H^2

$$K(t|A_2) = \frac{e^{-t\alpha^2}}{4\pi t} \int_0^\infty \frac{\pi e^{-tr^2}}{\cosh^2 \pi r} dr = \frac{e^{-t\alpha^2}}{4\pi t} \sum_{n=0}^{\infty} \frac{B_{2n}}{n!} (2^{1-2n} - 1) t^n, \quad (\text{A.10})$$

$$\tilde{\zeta}(s|A_2) = \frac{\alpha^{2-2s}}{4\pi(s-1)} - \frac{1}{\pi} \int_0^\infty \frac{r (r^2 + \alpha^2)^{-s}}{1 + e^{2\pi r}} dr, \quad (\text{A.11})$$

$$\mathcal{F}_2^\beta = -\frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \int_0^\infty \frac{K_{3/2}(n\beta\sqrt{r^2 + \alpha^2}) (r^2 + \alpha^2)^{3/4}}{(n\beta)^{3/2} \cosh^2 \pi r} dr. \quad (\text{A.12})$$

In Eq. (A.10), B_n are the Bernoulli numbers and the series is convergent for $0 < t < 2\pi$.

Note that in the paper the relevant operator concerning hyperbolic metric is A_{N-2n} with $\alpha = 0$. In this limit, equations above notably simplify, especially in the odd N cases. For example, from Eq. (A.8) in the limit $\alpha \rightarrow 0$ we have

$$\mathcal{F}_3^\beta = -\frac{\zeta_R(4)}{\pi^2 \beta^4} = -\frac{\pi^2}{90 \beta^4}. \quad (\text{A.13})$$

A.2 Spheres

As has been shown in Ref. [51], the recurrence relations (A.3-A.5) are also valid for manifolds with positive constant curvature ($\kappa = 1$). This means that for spheres, heat-kernel, ζ -function and free energy can be obtained by the knowledge of the same quantities for S^1 and S^2 . In the paper we only need heat-kernel coefficients and ζ -function. For the circle S^1 we have the representation

$$K(t|A_1) = \frac{e^{-t\alpha^2}}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/t}, \quad (\text{A.14})$$

$$\tilde{\zeta}(s|A_1) = \frac{\Gamma(s - \frac{1}{2})\alpha^{1-2s}}{\sqrt{4\pi}\Gamma(s)} + \frac{2\alpha^{1-2s}\sin\pi s}{\pi} \int_1^\infty \frac{(r^2 - 1)^{-s}}{e^{2\pi\alpha r} - 1} dr, \quad (\text{A.15})$$

while for S^2

$$K(t|A_2) = \frac{e^{-t\alpha^2}}{8it} \int_\Gamma \frac{e^{-tz^2}}{\cos^2 \pi z} dz, \quad (\text{A.16})$$

$$\tilde{\zeta}(s|A_2) = \frac{1}{8i(s-1)} \int_\Gamma \frac{(z^2 + \alpha^2)^{-(s-1)}}{\cos^2 \pi z} dz, \quad (\text{A.17})$$

where Γ is an open path in the complex plane going (clockwise) from ∞ to ∞ around the positive real axis enclosing the point $z = \varrho_2 = 1/2$ (see Ref. [51]). Note that in the paper the operator concerning spheres is L_{N-1} , so we have to choose $\alpha^2 = C - \varrho_{N-1}^2$. Of course, other representations are at disposal and can be found for example in Refs. [71, 54, 51]. For example,

$$\zeta(s|L_2) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(s+n) \left(C - \frac{1}{4}\right)^n}{\Gamma(n+1)\Gamma(s)} \zeta_H(2s+2n-1, 1/2), \quad (\text{A.18})$$

whence $\zeta(0|L_2) = C - 1/3$.

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